Turbulent transport and coherent structures in 3D plasmas

Keith Leitmeyer

April 21, 2015

Abstract

Kinetic and magnetic enstrophy are shown to concentrate to smaller scales from the integral scale to a Kraichnan-type scale for the 3D magnetohydrodynamic equations. This is an improvement of the result from Bradshaw and Grujić (2013), using redesigned ensemble averages.

1 Introduction

Observational and numerical evidence suggests that in magnetohydrodynamic (MHD) turbulence the vorticity $\omega = \nabla \times u$ and current $j = \nabla \times b$ (here, u and b are the velocity and the magnetic field, respectively) concentrate on coherent quasi low-dimensional structures–predominantly quasi two-dimensional sheets–which become increasingly thin exhibiting morphological dynamics consistent with a process known as *turbulent cascade*. Understanding this process is important due to its intimate relationship with violent reconnection events in the realm of *solar wind turbulence*, and is highly relevant to the goals outlined in the Space Studies Board of the National Research Council (NRC) survey, 22*A Decadal Strategy for Solar and Space Physics (Heliophysics)*, completed in 2012, and in particular, to the strategic goal to "discover and characterize fundamental processes that occur both within the heliosphere and throughout the universe"[17].

As a matter of fact–in the last five years–there has been a flurry of activity in the heliophysics community directed at understanding the phenomenon of *turbulent dissipation*, especially within the range of kinetic scales. One of the most promising theories, supported by extensive computational simulations, is that the coherent structures (most notably *current sheets*) exhibit a process of turbulent cascade down to the kinetic scales, essentially, all the way to electron scales, where they trigger extremely strong and *localized* heating of the plasma, dissipating the energy (the so-called *heating via current sheets*)[14, 22, 23]. There is also a sense of optimism in part of the community that the relevance of this theory to the solar wind could be confirmed via the data to be collected by the upcoming NASA Magnetospheric Multiscale (MMS) mission.

The zero-step in this theory is an assumption on the geometry of the turbulent plasma at the *interface* between the continuum (described by the MHD system) and the kinetic (described by the Vlasov-Maxwell-Poisson system)-scale dynamics; shortly, the predominance

of the current sheet geometry originating at continuum scales is *assumed*, and then utilized as the input when descending into the kinetic-scale dynamics. This brings out the question of *existence* of *turbulent cascades*–and in particular, the cascades of kinetic and magnetic enstrophies–in the *continuum*/MHD regime to the forefront of scientific interest.

Kinetic and magnetic enstrophy was shown to concentrate towards smaller scales in [1] using ensemble averages in physical scales. Here the ensemble averages used are redesigned to allow weaker assumptions. The ensemble averages used to state the enstrophy concentration theorem are described first. The following section goes over the conditions under which enstrophy concentration is shown. Then we recall the bounds obtained in [1], and finally the enstrophy concentration theorem is proven.

Related work on cascades and locality in hydrodynamic turbulence can be found in [2-5, 7-13, 15, 16, 18-21].

2 Fluxes and ensemble averages

The MHD equations, which model evolution of the velocity and magnetic fields in an electrically conducting incompressible fluid, read

$$\begin{aligned} &\partial_t u - \Delta u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla P = 0,\\ &\partial_t b - \Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0,\\ &\nabla\cdot u = \nabla\cdot b = 0. \end{aligned} \tag{1}$$

(Here, P is the total pressure, and the magnetic resistivity and the kinematic viscosity are normalized to 1.)

Taking the curl of equation 1 gives equations for vorticity and current,

$$\begin{aligned} &\partial_t\omega - \Delta\omega = -(u\cdot\nabla)\omega + (\omega\cdot\nabla)u + (b\cdot\nabla)j - (j\cdot\nabla)b,\\ &\partial_t j - \Delta j = -(u\cdot\nabla)j + (j\cdot\nabla)u + (b\cdot\nabla)\omega - (\omega\cdot\nabla)b + 2\sum_{l=1}^{3}\nabla b_l\times\nabla u_l. \end{aligned} \tag{2}$$

These equations can be used to study *inward* kinetic and magnetic enstrophy fluxes. Rather than traditional fluxes through the boundary of a ball $B = B(x_0, 2R)$,

$$\begin{aligned} -\int_{\partial B}\frac{1}{2}|\omega|^2(u\cdot n)\,d\sigma &= -\int_B (u\cdot\nabla)\omega\cdot\omega\,dx,\\ -\int_{\partial B}\frac{1}{2}|j|^2(u\cdot n)\,d\sigma &= -\int_B (u\cdot\nabla)j\cdot j\,dx, \end{aligned} \tag{3}$$

we use a smooth cutoff function ψ supported on $B(x_0, 2R)$ and equal to 1 on $B(x_0, R)$– with inward pointing gradient–and study fluxes through the spherical layer of thickness R,

$$\begin{aligned} \int\frac{1}{2}|\omega|^2(u\cdot\nabla\psi)\,dx &= -\int(u\cdot\nabla)\omega\cdot\psi\omega\,dx,\\ \int\frac{1}{2}|j|^2(u\cdot\nabla\psi)\,dx &= -\int(u\cdot\nabla)j\cdot\psi j\,dx. \end{aligned} \tag{4}$$

The reason is that this form of the flux is more amenable to mathematical analysis, while at the same time preserving the physics. Time-averaged quantities are studied in turbulence, so we take time averages weighted according to a smooth function $\eta(t)$ which is 0 on $[0, T/3]$ and 1 on $[2T/3, T]$, denoting $\phi(x,t) = \psi(x)\eta(t)$:

$$
\begin{aligned}
\int_0^T \int \frac{1}{2}|\omega|^2 (u \cdot \nabla \phi)\, dx\, dt &= -\int_0^T \int (u \cdot \nabla)\omega \cdot \phi\omega\, dx\, dt, \\
\int_0^T \int \frac{1}{2}|j|^2 (u \cdot \nabla \phi)\, dx\, dt &= -\int_0^T \int (u \cdot \nabla) j \cdot \phi j\, dx\, dt.
\end{aligned} \tag{5}
$$

Multiplying equation 2 by $\phi\omega$ and ϕj respectively, and integrating over space and time, will yield expressions which can be used to *dynamically* estimate the fluxes:

$$
\begin{aligned}
&\int_0^T \int \frac{1}{2}|\omega|^2 (u \cdot \nabla \phi)\, dx\, dt = \int \frac{1}{2}|\omega(x,T)|^2 \psi(x)\, dx + \int_0^T \int |\nabla \omega|^2 \phi\, dx\, dt \\
&- \int_0^T \int \frac{1}{2}|\omega|^2 (\partial_s \phi + \Delta \phi)\, dx\, dt - \int_0^T \int (\omega \cdot \nabla) u \cdot (\phi\omega)\, dx\, dt \\
&- \int_0^T \int (b \cdot \nabla) j \cdot (\phi\omega)\, dx\, dt + \int_0^T \int (j \cdot \nabla) b \cdot (\phi\omega)\, dx\, dt \\
&= \int \frac{1}{2}|\omega(x,T)|^2 \psi(x)\, dx + \int_0^T \int |\nabla \omega|^2 \phi\, dx\, dt + H^\omega + N_1^\omega + L^\omega + N_2^\omega
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&\int_0^T \int \frac{1}{2}|j|^2 (u \cdot \nabla \phi)\, dx\, dt = \int \frac{1}{2}|j(x,T)|^2 \psi(x)\, dx + \int_0^T \int |\nabla j|^2 \phi\, dx\, dt \\
&- \int_0^T \int \frac{1}{2}|j|^2 (\partial_s \phi + \Delta \phi)\, dx\, dt + \int_0^T \int (\omega \cdot \nabla) b \cdot (\phi j)\, dx\, dt \\
&- \int_0^T \int (b \cdot \nabla) \omega \cdot (\phi j)\, dx\, dt - \int_0^T \int (j \cdot \nabla) u \cdot (\phi j)\, dx\, dt \\
&- \int_0^T \int \left(2 \sum_{l=1}^{3} \nabla u_l \times \nabla b_l \right) \cdot (\phi j)\, dx\, dt \\
&= \int \frac{1}{2}|j(x,T)|^2 \psi(x)\, dx + \int_0^T \int |\nabla j|^2 \phi\, dx\, dt + H^j + N_1^j + L^j + N_2^j + X
\end{aligned} \tag{7}
$$

Enstrophy concentration will be demonstrated locally, over a ball $B(0, 2R_0)$. Ultimately we want to show that the average of these fluxes over suitable collections of functions ψ of a particular scale is positive, for a range of scales.

Fix $C_0 > 1$ and $3/4 < \rho < 1$. A **refined test function** at scale R is any smooth function ψ supported in a ball of radius $2R$ satisfying $0 \le \psi \le 1$, $|\nabla \psi| < \frac{C_0}{R}\psi^\rho$, and $|\Delta \psi| < \frac{C_0}{R^2}\psi^{2\rho - 1}$.

Now fix a scale R_0 refined test function ψ_0 centered at 0. An **ensemble** at scale R with global multiplicity K_1 and local multiplicity K_2 is a collection of scale R test functions $\{\psi_i\}_{i=1}^n$ satisfying the following properties:

1. $\psi_i \le \psi_0 \le \sum \psi_i$
2. $(R_0/R)^3 \le n \le K_1 (R_0/R)^3$

3. No point of $B(2R_0, 0)$ is contained in more than K_2 of the supports of ψ_i.

For a function f, denote by $\langle F\rangle_R$ the **ensemble average** $\frac{1}{n}\sum_{i=1}^n \frac{1}{R^3}\int f\psi_{i,R}\,dx$, and $F_0 = \frac{1}{R_0^3}\int f\psi_0\,dx$.

Property 1 above is needed to compare $\langle F\rangle_R$ to F_0. Due to Property 1, test functions near the boundary of the support of ψ_0 will have small integrals, effectively skewing the ensemble average towards zero. Larger K_1 and K_2 allow ensembles that have higher weight on functions away from the boundary, making the skewing insignificant.

These ensemble averages can be viewed as a way to detect whether a function is significantly negative above some spatial scale. If every ensemble average $\langle F\rangle_R$ is positive, no matter how one arranges and stacks the test functions, then the function is not significantly negative at scales larger than R. Increasing K_1 and K_2 lowers the threshold for a function to be considered significantly negative.

One way to explicitly construct ensembles is to apply Lemma 2 to ψ_0 and varying the multiplicity of the resulting functions (Assume ψ_0 satisfies the stronger C_0'-bounds to get an ensemble with C_0-bounds).

The following lemma states that ensemble averages (at any scale) of positive functions are comparable to the large scale mean. The proof immediately follows from the definitions.

Lemma 1. *If $f \geq 0$ then $\frac{1}{K_1}F_0 \leq \langle F\rangle_R \leq K_2 F_0$. For slightly modified ensemble averages, we have $\frac{1}{n}\sum_1^n \frac{1}{R^3}\int f\psi_{i,R}^\delta\,dx \leq K_2\frac{1}{R_0^3}\int f\psi_0^\delta\,dx$ $(\delta > 0)$.* □

Using a refined partition of unity, one can turn larger scale ensembles into smaller scale ensembles.

Lemma 2. *Any scale R test function satisfying C_0 bounds is a sum of $8\lceil R/R'\rceil^3$ scale R' test functions satisfying C_0' bounds (where $R > R'$ and C_0' depends only on C_0).*

Therefore for all (K_1, K_2, C_0)-ensembles at scale R and every $R' < R$, there exists a $(64K_1, 8K_2, C_0')$-ensemble at scale R' such that $\langle F\rangle_R = \langle F\rangle_{R'}$.

Proof. Let ψ be a scale R test function satisfying C_0 bounds . Now to construct the partition of unity, take a scale R' test function g_0 (satisfying C_0 bounds), centered at zero and equal to 1 on $[-R', R']^3$. Define $g_p = g_0(x - 2R'p)$, where $p \in \mathbb{Z}^3$. Then $1 \leq \sum_p g_p \leq 2$ so we may define $h_p = g_p/\sum_q g_q$.

Some calculus shows that $|\nabla h_p| < \frac{6C_0}{R'}h_p^\rho$ and $|\Delta h_p| < \frac{3C_0+10C_0^2}{R'^2}h_p^{2\rho-1}$, so $|\nabla(\psi h_p)| < \frac{7C_0}{R'}(\psi h_p)^\rho$ and $|\Delta(\psi h_p)| < \frac{4C_0+22C_0^2}{R'^2}(\psi h_p)^{2\rho-1}$. Fewer than $8\lceil R/R'\rceil^3 \leq 64(R/R')^3$ of the functions ψh_p are nonzero, and for any x, $\psi_p(x) \neq 0$ for at most 8 functions.

Since $\psi = \sum_p \psi h_p$, the first claim is proven. For the second claim, given an ensemble $\{\psi_i\}_i$, the new ensemble will be $\{\psi_i h_p\}_{i,p}$. □

3 Assumptions

Now we come to the conditions on the current and vorticity over $B(0, 2R_0) \times (0, T)$ under which we can show that there is enstrophy concentration.

3.1 Geometry/smoothness

Denote by $\theta(u,v)$ the angle between two vectors u, v. It is required that for some $M, C_1 > 0$,

$$|\sin\theta(\omega(x+y,t),\omega(x,t))| \leq C_1|y|^{1/2} \tag{8}$$

for every $t \in (0,T)$, every $x \in B(0, 2R_0 + R_0^{2/3})$ with $|\nabla u(x,t)| > M$, and every y such that $|y| < 2(\sigma_0/\beta) + (\sigma_0/\beta)^{2/3}$, and

$$|j(x+y,t) - j(x,t)| \leq |j(x+y,t)||y|^{1/2} \tag{9}$$

for every $t \in (0,T)$, every $x \in B(0, 2R_0 + R_0^{2/3})$ with $|\nabla b(x,t)| > M$, and every y such that $|y| < 2(\sigma_0/\beta) + (\sigma_0/\beta)^{2/3}$. The quantity σ_0/β is defined in section 3.2.

Condition 8 depends only on the angle of the vorticity vector ω. This $\frac{1}{2}$–Holder coherence will deplete the vortex stretching term. This condition is needed because condition 9 is insufficient to control the vortex stretching term N_1^ω, which has no explicit dependence on the magnetic field.

Condition 9 is needed for the nonlinear terms that do not have a geometric kernel available. It requires $\frac{1}{2}$–Holder continuity, and more. It would be too restrictive if $|j(x+y,t)|$ were ever small, but $x+y$ is always close to the region where $|\nabla b|$ is large, and roughly, ∇b and j are large in the same regions.

In [1], condition 9 was used with j replaced by ω with no assumption corresponding to 8. The present formulation is preferred because current appears to be more regular than vorticity in numerical simulations.

3.2 Kraichnan-type scale

Let e_0, E_0, and P_0 denote the time-averaged total energy, total enstrophy, and total palinstrophy at the integral scale. Precisely,

$$\begin{aligned}
e_0 &= \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0^{4\rho-3}\left(\frac{|u|^2}{2} + \frac{|b|^2}{2}\right) dx\, dt, \\
E_0 &= \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0^{2\rho-1}(|\omega|^2 + |j|^2)\, dx\, dt, \\
P_0 &= \frac{1}{T}\int_0^T \frac{1}{R_0^3}\int \phi_0(|\nabla\omega|^2 + |\nabla j|^2)\, dx\, dt + \frac{1}{TR_0^3}\int \frac{1}{2}(|\omega(x,T)|^2 + |j(x,T)|^2)\psi_0\, dx.
\end{aligned}$$

Define the modified Kraichnan-type scale σ_0 by

$$\sigma_0 = \max\{\left(\frac{E_0}{P_0}\right)^{1/2}, \left(\frac{e_0}{P_0}\right)^{1/4}\}. \tag{10}$$

Assumption 2 is that $\sigma_0 < \beta R_0$, where β is a constant ($0 < \beta < 1$) identified in the proof. If ω and j have large gradients as expected in a turbulent flow, this assumption will be satisfied.

3.3 Localization

Any smooth solution to the equations will have $\omega \in L^2((0,T) \times B(0, 2R_0 + R_0^{2/3}))$. It is required that the kinetic and magnetic enstrophy is not too highly concentrated around any point:

$$\int_0^T \int_{B(y,R)} |\omega|^2 + |j|^2 \, dx \, dt < \frac{1}{C_2} \tag{11}$$

for any $y \in B(0, 2R_0)$ where C_1 is a constant and $R = 2\sigma_0/\beta + (\sigma_0/\beta)^{2/3}$.

In [1], it was required that

$$\int_0^T \int_{B(0,2R_0+R_0^{2/3})} |\omega|^2 \, dx \, dt < \frac{1}{C_3}.$$

This assumption restricts the extent of the range of scales over which we can show enstrophy concentration. The magnetic enstrophy is in the present assumption because of the geometric/smoothness assumption on the current that is not found in [1].

3.4 Modulation

The assumption imposes a restriction on the time evolution of the integral-scale kinetic and magnetic enstrophies across $(0, T)$ consistent with our choice of the temporal cutoff. Precisely,

$$\int |\omega(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_t \int |\omega(x,t)|^2 \psi_0(x) \, dx,$$
$$\int |j(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_t \int |j(x,t)|^2 \psi_0(x) \, dx,$$

4 Bounds

In [1], the terms of equations 6 and 7 are bounded by quantities which can be related to e_0, E_0, and P_0.

Obtaining the desired bounds using Assumption 3.1 will use all of the same techniques as in [1] with the major difference being labelling, except for the vortex stretching term which uses geometric depletion as in [6].

Ultimately, we have

$$\begin{aligned} &H^\omega + H^j + N_1^\omega + N_2^\omega + N_1^j + N_2^j + L^\omega + L^j + X \\ &\quad \leq K_P \Big(\frac{1}{\alpha} + ||\omega||_{L^2((0,T)\times B(x_i, 2R+R^{2/3}))} \Big) \Big(\frac{1}{2} \sup_{t\in(0,T)} \int \psi(x)(|\omega(x,t)|^2 + |j(x,t)|^2) \, dx + \\ &\quad \int_0^T \int \phi(|\nabla \omega|^2 + |\nabla j|^2) \, dx \, dt \Big) + \frac{K_E}{R^2} \int_0^T \int \phi^{2\rho-1}(|\omega|^2 + |j|^2) \, dx \, dt + \\ &\quad \frac{\alpha^2 K_e}{R^4} \int_0^T \int \phi^{4\rho-3} \frac{|u|^2 + |b|^2}{2} \, dx \, dt, \end{aligned}$$

where K_P, K_E, K_e are constants and α is an interpolation parameter that we may choose. The constants K_P, K_E, K_e are fixed upon a choice of the parameters K_1, K_2, ρ, C_0, C_1, and M. In the following section K_P, K_E, K_e are instead the constants obtained corresponding to $64K_1, 8K_2$, and C_0'.

5 Main result

Kinetic and magnetic enstrophy is on average being transported to smaller scales down to the Kraichnan-type scale σ_0/β. The following notation will be used in the proof: For a density f and ensemble $\{\psi_i\}_{i=1}^n$, the average over one element of the ensemble is $F_i = \int f\psi_i\,dx$ and the ensemble average is $\langle F\rangle_R = \frac{1}{n}\frac{1}{TR^3}\sum_{i=1}^n F_i$. Let $\varphi = -\int_0^T (u\cdot\nabla)\omega\cdot\omega + (u\cdot\nabla)j\cdot j\,dt$, the enstrophy flux density, so that $\Phi_i = \int \varphi\psi_i\,dx$ is the kinetic and magnetic enstrophy flux centered at x_i at scale R. Referring to equations 6 and 7, suppressing the subscript i, denote $H = H^\omega + H^j$ and likewise for N and L. Define similarly

$$e = \int_0^T \int \phi_i^{4\rho-3}\frac{1}{2}(|u|^2+|b|^2)\,dt\,dt,$$

$$E = \int_0^T \int \phi_i^{2\rho-1}(|\omega|^2+|j|^2)\,dx\,dt,$$

$$P = \int \psi_i(x)(|\omega(x,T)|^2+|j(x,T)|^2)\,dx + \int_0^T \int \phi_i(|\nabla\omega|^2+|\nabla j|^2)\,dx\,dt,$$

and

$$\tilde{P} = \frac{1}{2}\sup_{t\in(0,T)} \int \psi_i(x)(|\omega(x,t)|^2+|j(x,t)|^2)\,dx + \int_0^T \int \phi_i(|\nabla\omega|^2+|\nabla j|^2)\,dx\,dt.$$

Theorem. *Under Assumptions 1-4, for any K_1 and K_2 there exists K_* such that for any (K_1,K_2)-ensemble at scale R ranging from σ_0/β to R_0, we have* $\frac{1}{K_*}P_0 \le \langle\Phi\rangle_R \le K_*P_0$.

Proof. We start by showing that for $R = \sigma_0/\beta$, for $(64K_1, 8K_2)$-ensembles satisfying C_0' bounds, we have $\frac{1}{K_*}P_0 \le \langle\Phi\rangle_R \le K_*P_0$.

Referring to equations 6 and 7, $\langle\Phi\rangle_R = \langle P\rangle_R + \langle H+N+L+X\rangle_R$, where p is the positive density $\frac{1}{2}|\omega(T)|^2 + \int_0^T |\nabla\omega|^2\eta\,dt$ so that $\frac{1}{64K_1}P_0 \le \langle P\rangle_R \le 8K_2P_0$. The rest of the terms are relatively small upon averaging: $|H+N+L+X| \le K_P\Big(\frac{1}{\alpha} + ||\omega||_{L^2(B(x_i,2R+R^{2/3})\times(0,T))}\Big)\tilde{P} +$

$\dfrac{K_E}{R^2}E + \dfrac{\alpha^2 K_e}{R^4}e$, so that

$$
\begin{aligned}
|\langle H+N+L+X\rangle_R| &\leq \Big\langle K_P\big(\frac{1}{\alpha}+||\omega||\big)\tilde{P} + \frac{K_E}{R^2}E + \frac{\alpha^2 K_e}{R^4}e\Big\rangle_R \\
&\leq 8K_2K_P\Big(\frac{1}{\alpha}+||\omega||\Big)\tilde{P}_0 + \frac{8K_2K_E}{R^2}E_0 + \frac{\alpha^2 8K_2K_e}{R^4}e_0 \\
&= \Big(8K_2K_P\big(\frac{1}{\alpha}+||\omega||\big) + 8K_2K_E\beta^2 + \alpha^2 8K_2K_e\beta^4\Big)P_0 \\
&\leq \frac{3}{4\cdot 64K_1}P_0,
\end{aligned}
$$

by taking $\alpha = 4\cdot 64K_1\cdot 8K_2K_P$, using that β is sufficiently small such that $8K_2K_E\beta^2 + \alpha^2\cdot 8K_2K_e\beta^4 \leq \frac{1}{4\cdot 64K_1}$, and using the assumption that $||\omega||_{L^2(B(x_i,2R+R^{2/3})\times(0,T))} \leq \frac{1}{C_2} = \frac{1}{4\cdot 64K_1\cdot 8K_2K_P}$.

Therefore $\dfrac{1}{4\cdot 64K_1}P_0 \leq \langle\Phi\rangle_R \leq (8K_2 + \dfrac{3}{4\cdot 64K_1})P_0$ for any $(64K_1, 8K_2, C_0')$-ensemble at scale $R = \sigma_0/\beta$. For the range of scales from σ_0/β up to R_0, by Lemma 2, $\dfrac{1}{4\cdot 64K_1}P_0 \leq \langle\Phi\rangle_R \leq (8K_2 + \dfrac{3}{4\cdot 64K_1})P_0$ for all scale R (K_1, K_2, C_0)-ensembles.

□

The locality of magnetic and kinetic enstrophy flux stated in [1] also holds in our setting of more satisfying assumptions and modified ensemble averages. The result is stated in terms of the time-averaged enstrophy flux, rather than the time-averaged enstrophy flux per unit mass used above. That is,

$$
\langle\Psi\rangle_R = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\int_0^T\int \frac{1}{2}(|\omega|^2+|j|^2)(u\cdot\nabla\phi_i)\,dx\,dt
$$

for an ensemble $\{\phi_i\}_{i=1}^n$ at scale R.

Corollary. *Under Assumptions 1-4, for any K_1, K_2 there exists K_* such that for any r, R between σ_0/β and R_0 and any (K_1, K_2) ensembles, enstrophy flux is local:*

$$
\frac{1}{K_*^2}\Big(\frac{r}{R}\Big)^3 \leq \frac{\langle\Psi\rangle_r}{\langle\Psi\rangle_R} \leq K_*^2\Big(\frac{r}{R}\Big)^3.
$$

Acknowledgments

I would like to thank my advisor, Professor Grujić, for his suggestions and guidance.

References

[1] Z. Bradshaw and Z. Grujić. On the transport and concentration of enstrophy in 3D magnetohydrodynamic turbulence. *Nonlinearity*, 26:2373, 2013.

[2] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy. Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity*, 21, 2008.

[3] P. Constantin. Navier-Stokes equations and area of interfaces. *Commun. Math. Phys.*, 129:241, 1990.

[4] P. Constantin. Geometric statistics in turbulence. *SIAM Rev.*, 36, 1994.

[5] R. Dascaliuc and Z. Grujić. Energy cascades and flux locality in physical scales of the 3D Navier-Stokes equations. *Commun. Math. Phys.*, 305:199, 2011.

[6] R. Dascaliuc and Z. Grujić. Coherent vortex structures and 3D enstrophy cascade. *Commun. Math. Phys.*, 317:547, 2013.

[7] G. Eyink. Locality of turbulent cascades. *Physica D*, 207, 2005.

[8] G. Eyink and H. Aluie. Localness of energy cascade in hydrodynamic turbulence. i. Smooth coarse graining. *Phys. Fluids*, 21, 2009.

[9] C. Foias, M.S. Jolly, O. Manley, and R. Rosa. Statistical estimates for the Navier-Stokes equations and the Kraichnan theory of 2-D fully developed turbulence. *J. Stat. Phys.*, 102, 2005.

[10] C. Foias, O. Manley, R. Rosa, and R. Temam. Estimates for the energy cascade in three-dimensional turbulent flows. *C.R. Acad. Sci. Paris Sér. I Math*, 333, 2001.

[11] Z. Grujić. Vortex stretching and anisotropic diffusion in 3D Navier-Stokes equations. to appear in *Contemporary Mathematics*.

[12] Z. Grujić. Localization and geometric depletion of vortex-stretching in the 3D NSE. *Commun. Math. Phys.*, 290:861, 2009.

[13] J. J. Jimenez, A.A. Wray, P.G. Saffman, and R.S. Rogallo. The structure of intense vorticity in isotropic turbulence. *J. Fluid Mech.*, 255, 1993.

[14] Karimabadi, Roytershteyn, Wan, Matthaeus, and Daughton. Coherent structures, intermittent turbulence, and dissipation in high-temperature plasmas. *Phys. Plasmas*, 20, 2013.

[15] K. Leitmeyer. Enstrophy cascade in physical scales for the 3D Navier-Stokes equations. *ArXiv e-prints*, February 2015. http://arxiv.org/abs/1502.01258.

[16] V. L'vov and G. Falkovich. Counterbalanced interaction locality of developed hydrodynamic turbulence. *Phys. Rev. A*, 46, 1992.

[17] A decadal strategy for solar and space physics (heliophysics), http://sites.nationalacademies.org/ssb/currentprojects/ssb_056864.

[18] R. Rosa. Some results on the Navier-Stokes equations in connection with the statistical theory of stationary turbulence. *Applications of Mathematics*, 47:485, 2002.

[19] Z.-S. She, E. Jackson, and S. Orszag. Structure and dynamics of homogeneous turbulence: models and simulations. *Proc. R. Soc. Lond. A*, 434, 1991.

[20] G.I. Taylor. Production and dissipation of vorticity in a turbulent fluid. *Proc. Roy. Soc*, 1937.

[21] A. Vincent and M. Meneguzzi. The dynamics of vorticity tubes in homogeneous turbulence. *J. Fluid Mech.*, 225, 1994.

[22] Yoshimatsu, Okamoto, Kawahara, Schneider, and Farge. Coherent vorticity and current density simulation of three-dimensional magnetohydrodynamic turbulence using orthogonal wavelets. *Geophysical and Astrophysical Fluid Dynamics*, 107:73, 2013.

[23] Zhdankin, Uzdensky, Perez, and Boldyrev. Statistical analysis of current sheets in three-dimensional magnetohydrodynamic turbulence. *The Astrophysical Journal*, 771, 2013.